

PREPARED FOR SUBMISSION TO JHEP

Modifications to Holographic Entanglement Entropy in Warped CFT

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ABSTRACT: In [1] it was observed that asymptotic boundary conditions play an important role in the study of holographic entanglement beyond AdS/CFT. In particular, the Ryu-Takayanagi proposal must be modified for warped AdS₃ (WAdS₃) with Dirichlet boundary conditions. In this paper, we consider AdS₃ and WAdS₃ with Dirichlet-Neumann boundary conditions. The conjectured holographic duals are warped conformal field theories (WCFTs), featuring a Virasoro-Kac-Moody algebra. We provide a holographic calculation of the entanglement entropy and Rényi entropy using AdS₃/WCFT and WAdS₃/WCFT dualities. Our bulk results are consistent with the WCFT results derived by Castro-Hofman-Iqbal using the Rindler method. Comparing with [1], we explicitly show that the holographic entanglement entropy is indeed affected by boundary conditions. Both results differ from the Ryu-Takayanagi proposal, indicating new relations between spacetime geometry and quantum entanglement for holographic dualities beyond AdS/CFT.

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1 Introduction

The motivation of studying holography beyond AdS/CFT [2] comes both from the bulk side and the boundary side. In the bulk, to understand quantum gravity in the real world, it is necessary to consider spacetimes without being asymptotic to locally AdS. Progresses include dS/CFT correspondence [3], the Kerr/CFT correspondence [4], the WAdS/CFT [5] or WAdS/WCFT [6] correspondence, flat space holography [7], BMS group [8], etc. On the boundary, there are richer physics beyond the critical point. Developments include Shrödinger or Lifshitz spacetime/non-relativistic field theory duality [9].

A three dimensional warped Anti-de Sitter space (WAdS₃), named and classified in [5], is a simple deformation of AdS₃, with $SL(2, R) \times U(1)$ isometry. Due to a drastically different asymptotic behaviour, a novel type of holographic dual, if exists, is expected. So far there are two versions of conjectures for the holographic duality, namely, the so-called WAdS₃/CFT [5], and the WAdS₃/WCFT correspondence [6].

The WAdS₃/CFT₂ conjecture was motivated by the observation that the Bekenstein-Hawking entropy of WAdS₃ black holes can be captured by Cardy's formula [5, 10]. In support of the conjecture, a Dirichlet type of boundary conditions were found in [11] under which the asymptotic symmetry group is generated by two Virasoro algebra.

The WAdS₃/WCFT conjecture is based on Dirichlet-Neumann boundary conditions [12]. The so-called two dimensional warped conformal field theory (WCFT) is defined by a warped conformal symmetry generated by a Virasoro-Kac-Moody algebra [6]. WCFTs are usually not Lorentzian invariant. Much like the derivation of the Cardy formula [13] for a CFT, modular transformations lead to a Detournay-Hartman-Hofman(DHH) formula [6] for the thermal entropy of WCFTs. The agreement between the WAdS black hole entropy and the microscopic entropy calculated by the DHH formula gives evidence to this WAdS/WCFT holography. See also [14] for a pure field theoretic discussion of the symmetries, [15, 16] for explicit examples, and [17] for partition functions of WCFTs. Furthermore, it was found in [18] that Dirichlet-Neumann boundary conditions, later called the Compère-Song-Strominger (CSS) boundary conditions can also be imposed to AdS₃. This leads to an alternative holographic dual to AdS₃, namely the AdS₃/WCFT correspondence. WAdS₃ can be embedded into string theory [10, 19–22]. In some examples [20–22], thermodynamics of WAdS₃ are even identical to those of AdS₃. Therefore AdS₃/WCFT can be used as a good starting point of studying WAdS₃/WCFT.

So far either the framework of WAdS₃/CFT or WAdS₃/WCFT provides a microscopic explanation of the Bekenstein-Hawking entropy of WAdS₃ black holes. As a finer probe, entanglement entropy is expected to further test these two holographic dualities, which is the main focus of this paper.

In the context of AdS/CFT, the seminal work of Ryu and Takayanagi [23, 24] provides a powerful tool to understand the relationship between spacetime geometry and quantum entanglement. Ryu-Takayanagi proposal is now firmly established [25–28] in the context of Einstein gravity, with a static, asymptotically AdS spacetime in the bulk. For WAdS₃, it is interesting to ask whether the Ryu-Takayanagi formula or the covariant HRT formula [29] is still valid, and how to derive/prove it if the answer is yes; or what is the analog if

the answer is no. The first attempt appeared in [30], where a tension was found between the HRT formula and the WAdS/CFT duality. See also [31] for more discussions along this line. Recently the holographic entanglement entropy for WAdS₃ spacetime with Dirichlet boundary conditions was calculated in [1] with a modified Lewkowycz-Maldacena [28] prescription. The result is consistent with the WAdS/CFT duality, but is different from a direct use of the HRT formula. A key observation is that asymptotic boundary conditions play an important role in such holographic dualities beyond AdS/CFT. In this paper we consider AdS and WAdS with the Dirichlet-Neumann boundary conditions.

We study the holographic entanglement entropy in the context of AdS/WCFT and WAdS/WCFT correspondence. The approach we take is the Rindler method [25], which was developed in the context of AdS/CFT. On the CFT side, a certain conformal transformation maps the calculation of an entanglement entropy to the thermal entropy of a Rindler or hyperbolic space. Using the dictionary of AdS/CFT, the bulk counterpart of this procedure is to perform a coordinate transformation, mapping a certain region of Poincaré AdS to a hyperbolic black hole. The Bekenstein-Hawking entropy of the hyperbolic black hole then calculates the entanglement entropy holographically. Generalization to WCFT was carried out in [32], where the role of conformal transformations is now played by warped conformal transformations allowed by the symmetry of WCFT. The current paper provides a holographic calculation on the gravity side¹. Using the strategy of doing quotient on WAdS spacetime, we find that the analog of hyperbolic black holes are the WAdS black strings. We proposed that the thermal entropy of the WAdS black string gives the holographic calculation of entanglement entropy of WCFT. Explicitly agreement with the field theory result is found using the dictionary of AdS/WCFT and WAdS/WCFT. Rényi entropy in the bulk AdS₃ is also calculated and showed to be consistent with the WCFT calculation.

To summarise, we provide a holographic calculation of entanglement entropy and Rényi entropy for AdS and WAdS under the Dirichlet-Neumann boundary conditions. Our bulk results agree with those of WCFT [32], further supporting the conjectured AdS/WCFT and WAdS/WCFT dualities. Holographic entanglement entropy for WAdS₃ in this paper differs from that of [1], the reason of which is the different choices of asymptotic boundary conditions. Both this paper and [1] differ from the Ryu-Takayanagi proposal, indicating new relations between spacetime geometry and quantum entanglement for holographic dualities beyond AdS/CFT.

The layout of this paper is the following. In section 2, we briefly review WCFT, AdS₃/WCFT and WAdS₃/WCFT. In section 3 we re-calculate the entanglement entropy for WCFT, along the lines of [32]. In section 4, a bulk calculation is proposed. Then we match the results calculated on both sides in section 5. In section 6 we calculate the Rényi entropy, and show the agreement between the bulk and boundary calculations.

¹In [32], a bulk calculation was carried out for lower spin gravity [16], which does not have an usual geometric description

2 AdS₃/WCFT and WAdS₃/WCFT

In this section, we briefly review AdS₃/WCFT and WAdS₃/WCFT, and set up notations and conventions. In section 2.1 we list a few properties of WCFT. section 2.2 is for AdS₃/WCFT, and section 2.3 is for WAdS₃/WCFT.

2.1 WCFT

In this subsection, we discuss warped conformal field theory (WCFT) with a pure field theoretical setup. In [14] it was shown that a two dimensional local field theory with translational invariance $z = z' + z_0$, $w = w' + w_0$ and a chiral scaling symmetry $z = \gamma z'$ will have some enhanced symmetries. One minimal option is to have the following local symmetry

$$z = f(z'), \quad w = w' + g(z'). \quad (2.1)$$

The above property was later used as a definition of WCFT in [6]. On a cylinder, the conserved charges can be written in terms of Fourier modes. The WCFT algebra is [6]

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}, \\ [L_n, P_m] &= -mP_{n+m}, \\ [P_n, P_m] &= \frac{k}{2}n\delta_{n+m}, \end{aligned} \quad (2.2)$$

which describes a Virasoro algebra with central charge c and Kac-Moody algebra with level k , and furthermore the Kac-Moody generators transform canonically under the action of Virasoro generator. We will hereafter refer to (2.2) as the canonical WCFT algebra.

So far there are two concrete examples of WCFT. The chiral Liouville gravity [15] can be obtained from Chern-Simons formulation of Einstein gravity with CSS boundary conditions, analogous to the usual Liouville theory under Brown-Henneaux boundary conditions. The Weyl Fermion model [16] is holographically dual to the so-called lower spin gravity, whose partition function were calculated in [17].

2.2 AdS₃/WCFT

In this subsection we lay out a few properties of the AdS₃/WCFT correspondence, focusing on black holes and black strings. We start with the algebra of asymptotic symmetry under the CSS boundary conditions in section 2.2.1, and show how to relate it to canonical Virasoro-Kac-Moody algebra defining a WCFT in section 2.2.2. We showed how the entropy can be reproduced using the DHH formula. In section 2.2.3, we discuss black strings and show how to effectively calculate the thermodynamic quantities using a black hole.

2.2.1 AdS₃/WCFT_{(\hat{u}, \hat{v})}}

Consider BTZ black holes

$$ds^2 = \ell^2 \left(T_{\hat{u}}^2 d\hat{u}^2 + 2rd\hat{u}d\hat{v} + T_{\hat{v}}^2 d\hat{v}^2 + \frac{dr^2}{4(r^2 - T_{\hat{u}}^2 T_{\hat{v}}^2)} \right), \quad (2.3)$$

$$(\hat{u}, \hat{v}) \sim (\hat{u} + 2\pi, \hat{v} + 2\pi). \quad (2.4)$$

The local isometry is $SL(2, R) \times SL(2, R)$, while only the $U(1) \times U(1)$ part is globally well-defined due to the spacial circle (2.4). The Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi \ell}{2G} (T_{\hat{u}} + T_{\hat{v}}) . \quad (2.5)$$

Under the Brown-Henneaux boundary conditions, the asymptotic symmetry group for asymptotically AdS_3 spacetime is generated by two copies of Virasoro algebra. In [18], however, Dirichlet-Neumann boundary conditions, namely the CSS boundary conditions, for AdS_3 was found². Under the CSS boundary conditions the asymptotic symmetry group is generated by a Virasoro-Kac-Moody algebra. In this paper, we will focus on the CSS boundary conditions [18], under which Einstein gravity with negative cosmological constant on AdS_3 is conjectured to be due to a WCFT. The algebra of the asymptotic symmetry is given by

$$\begin{aligned} [\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} + \frac{\tilde{c}}{12}(n^3-n)\delta_{n+m}, \\ [\tilde{L}_n, \tilde{P}_m] &= -m\tilde{P}_{m+n} + m\tilde{P}_0\delta_{n+m}, \\ [\tilde{P}_n, \tilde{P}_m] &= \frac{\tilde{k}}{2}n\delta_{m+n}, \end{aligned} \quad (2.6)$$

with the central charge and Kac-Moody level

$$\tilde{c} = \frac{3\ell}{2G}, \quad \tilde{k} = 4\tilde{P}_0, \quad (2.7)$$

where the nonzero conserved charges for the BTZ metric (2.3) is given by

$$\tilde{P}_0 = -\frac{\ell T_{\hat{v}}^2}{4G}, \quad \tilde{L}_0 = \frac{\ell T_{\hat{u}}^2}{4G}. \quad (2.8)$$

As was argued in [6], the thermal entropy of a theory with the symmetry (2.6) can be written as a Cardy-like formula

$$S_{micro} = 2\pi\sqrt{-\tilde{P}_0^{vac}\tilde{P}_0} + 2\pi\sqrt{-\tilde{L}_0^{vac}\tilde{L}_0}, \quad (2.9)$$

where \tilde{P}_0^{vac} and \tilde{L}_0^{vac} are the vacuum values of the zero-mode charges. A natural way to find the vacuum charges is to rewrite the metric of BTZ black holes (2.3) in the Schwarzschild form, where it is easy to find that Global AdS_3 is the vacuum, with

$$T_{\hat{u}}^{vac} = \pm \frac{i}{2}, \quad T_{\hat{v}}^{vac} = \pm \frac{i}{2}, \quad (2.10)$$

or equivalently,

$$\tilde{P}_0^{vac} = \tilde{L}_0^{vac} = \frac{\tilde{c}}{24}. \quad (2.11)$$

²See also [34] for other choices of consistent boundary conditions.

Plugging these vacuum values into (2.9) the microscopic formula (2.9) matches the Bekenstein-Hawking entropy of the warped black string

$$S_{micro} = \frac{\pi \tilde{c}}{3} (T_{\hat{u}} + T_{\hat{v}}) = S_{BH}. \quad (2.12)$$

Note that the algebra (2.6) is different from the canonical WCFT algebra (2.2). To make distinctions, hereafter we will use the coordinates as subscripts. We denote the geometry (2.3) by $\text{AdS}_{(\hat{u}, \hat{v}, r)}$, and denote the field theory defined by the algebra (2.6) by $\text{WCFT}_{(\hat{u}, \hat{v})}$.

2.2.2 From $\text{WCFT}_{(\hat{u}, \hat{v})}$ to $\text{WCFT}_{(\hat{x}, \hat{t})}$

The algebra (2.6) and the canonical WCFT algebra (2.2) are related by a charge redefinition as was shown in [6]

$$\tilde{P}_n = \frac{2P_0 P_n}{k} - \frac{P_0^2 \delta_n}{k}, \quad \tilde{L}_n = L_n - \frac{2P_0 P_n}{k} + \frac{P_0^2 \delta_n}{k}. \quad (2.13)$$

with $c = \tilde{c}$. Note that none of the classical solutions (2.3) has a higher Kac-Moody hair, or in other words, they all satisfy $P_{n \neq 0} = 0$. For states with $P_{n \neq 0} = 0$, (2.13) amounts to a coordinate transformation,

$$\hat{u} = \hat{x}, \quad \hat{v} = \frac{k \hat{t}}{2P_0} + \hat{x}. \quad (2.14)$$

As discussed at the end of last subsection, we use $\text{WCFT}_{(\hat{x}, \hat{t})}$ to denote the WCFT on \hat{x}, \hat{t} .

For simplicity, we will always choose $k = -\frac{\ell}{G}$ throughout this paper, while other choices will not affect the discussion. Then it follows

$$P_0 = -\sqrt{\tilde{P}_0 k} = -\frac{\ell T_{\hat{v}}}{2G}, \quad L_0 = \tilde{L}_0 + \tilde{P}_0 = \frac{\ell}{4G} (T_{\hat{u}}^2 - T_{\hat{v}}^2). \quad (2.15)$$

The spacial circle (2.4) leads to a spacial circle in the \hat{x}, \hat{t} coordinates,

$$\text{spacial circle : } (\hat{x}, \hat{t}) \sim (\hat{x} + 2\pi, \hat{t}). \quad (2.16)$$

The thermal temperature of the black black hole is translated to a thermal circle of $\text{WCFT}_{(\hat{x}, \hat{t})}$

$$\text{thermal circle : } (\hat{x}, \hat{t}) \sim \left(\hat{x} + i \frac{\pi}{T_{\hat{u}}}, \hat{t} - i\pi \left(\frac{T_{\hat{v}}}{T_{\hat{u}}} + 1 \right) \right). \quad (2.17)$$

The vacuum expectation values for the zero modes (2.11) are then

$$P_0^{vac} = -\frac{i\ell}{4G}, \quad L_0^{vac} = 0. \quad (2.18)$$

Plugging the above into DHH formula [6] in this ensemble,

$$\begin{aligned} S_{DHH} &= -\frac{4\pi i P_0 P_0^{vac}}{k} + 4\pi \sqrt{-\left(L_0^{vac} - \frac{(P_0^{vac})^2}{k} \right) \left(L_0 - \frac{P_0^2}{k} \right)} \\ &= \frac{\pi c}{3} (T_{\hat{u}} + T_{\hat{v}}), \end{aligned} \quad (2.19)$$

we again reproduce the macroscopic entropy of the black hole, $S_{DHH} = S_{BH}$.

2.2.3 From $\text{WCFT}_{(u,v)}$ to $\text{WCFT}_{(\hat{u},\hat{v})}$

If we uncompactify the spacial circle (2.4), we get a BTZ black string with an infinite horizon

$$ds^2 = \ell^2 \left(T_u^2 du^2 + 2rdudv + T_v^2 dv^2 + \frac{dr^2}{4(r^2 - T_u^2 T_v^2)} \right). \quad (2.20)$$

Now consider the black string (2.20) on an arbitrary spatial interval

$$\{(u, v) | u = \Delta u(-\frac{1}{2} + \tau), \quad v = \Delta v(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}. \quad (2.21)$$

If the interval is very large, for example if $\Delta u \rightarrow \infty$, the system on (2.21) is equivalent to a black hole with the periodicity

$$(u, v) \sim (u + \Delta u, v + \Delta v). \quad (2.22)$$

In particular, the total charges, and the thermal entropy on the interval are always the same as if there is a spacial circle (2.4). As we will see later, we will always encounter the systems with $\Delta u \rightarrow \infty$. Hereafter, we will often treat theory on a large spacial interval as a spacial circle without further explanations. The total charges and entropy of the black hole (2.22)/black string (2.21) are the same as those of (2.3) (2.4) with the mapping

$$u = \frac{\Delta u}{2\pi} \hat{u}, \quad v = \frac{\Delta v}{2\pi} \hat{v}, \quad T_u = 2\pi \frac{T_{\hat{u}}}{\Delta u}, \quad T_v = 2\pi \frac{T_{\hat{v}}}{\Delta v}, \quad r = \frac{4\pi^2}{\Delta u \Delta v} \hat{r}, \quad (2.23)$$

Thus the thermal entropy is given by

$$S = \frac{\pi \ell}{2G} (\hat{T}_u + \hat{T}_v) = \frac{\ell}{4G} (\Delta u T_u + \Delta v T_v). \quad (2.24)$$

Correspondingly, the dual field theory derived from the asymptotic symmetry analysis are denoted by $\text{WCFT}_{(u,v)}$ and $\text{WCFT}_{(\hat{u},\hat{v})}$.

2.3 WAdS₃/WCFT

Warped AdS₃ (WAdS₃) appears in many context, including three dimensional theories [35–37], and some six dimensional theories [10], [20, 21], [38, 39], [22].

In this paper, we consider a class of locally WAdS₃ spacetimes, the warped black string solutions from consistent truncations of IIB string theory [22]. The metric can be written as

$$ds^2 = \ell^2 \left(T_u^2 (1 + \lambda^2 T_v^2) - \lambda^2 r^2 du^2 + 2rdudv + T_v^2 dv^2 + \frac{(1 + \lambda^2 T_v^2) dr^2}{4(r^2 - T_u^2 T_v^2)} \right). \quad (2.25)$$

The parameter λ is the warping parameter, whose existence breaks the $SL(2, R) \times SL(2, R)$ local isometry of AdS₃ to the $SL(2, R) \times U(1)$ local isometry of WAdS₃. When $\lambda = 0$, the warped black string metric goes back to the BTZ black string (2.20). A key feature of these models is that the thermodynamic properties of these warped black strings are independent of the warping factor λ . When $\lambda \neq 0$, we have to keep the spacial circle of (2.25) uncompactified, otherwise there will be closed time-like curves. However, similar to

BTZ black string, for the purposes of discussing the thermodynamics, calculations on a spacial interval

$$\{(u, v) | u = \Delta u(-\frac{1}{2} + \tau), \quad v = \Delta v(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}, \quad (2.26)$$

is effectively the same as on a spacial circle

$$(u, v) \sim (u + \Delta u, v + \Delta v). \quad (2.27)$$

With the Dirichlet-Neumann type of boundary conditions [11, 21], the asymptotic symmetry of (2.25) is generated by a chiral stress tensor and a U(1) current. On a cylinder, the symmetry is generated by the non-canonical Virasoro-Kac-Moody algebra (2.6). All the discussions in the previous subsection for AdS/WCFT can be repeated here in the context of WAdS/WCFT.

3 Entanglement entropy for WCFT

In this section we compute the entanglement entropy of a single interval in WCFT with an adapted version of the Rindler method [25], following the main steps of [32]. Note that this section is a purely field theoretical calculation, without any reference to holography.

Based on [40–43], [25] developed the Rindler method to derive the Ryu-Takayanagi formula for spherical entangling surfaces in the context of $\text{AdS}_{d+1}/\text{CFT}_d$. It contains both a field theory story and a gravity story. On the field theory side, a spherical entangling surface S^{d-2} at constant time slice is considered in the vacuum state of a d -dimensional CFT³. A certain conformal transformation maps the causal development of the subsystem to a Rindler space or a hyperbolic space, and maps the reduced density matrix of the former to a thermal density matrix of the later. Therefore the entanglement entropy is mapped to the thermal entropy of the Rindler/hyperbolic space. The gravity story is related to the field theory story by the AdS/CFT dictionary. A bulk extension of the conformal transformation can be found, which maps Poincaré AdS_{d+1} space to a hyperbolic black hole. The thermal entropy of the CFT is therefore the Bekenstein-Hawking entropy of the hyperbolic black hole.

The field theory story of the Rindler method [25] was extended to WCFT at any temperatures in [32]. There are three main differences. Firstly, as WCFT is not Lorentzian invariant, an interval is specified by both its length and its direction, and therefore is parameterized by two parameters. Similarly, both a thermal circle and a spatial circle is parameterized by two parameters. And thus a torus is parameterized by four parameters. Secondly, instead of a conformal transformation in CFT, warped conformal transformation in the form of (2.1) are the only allowed transformations in a WCFT. Therefore, a warped conformal transformation is needed to map a subregion to a space with a thermal identification. Similarly, the warped conformal transformation reduces the problem of calculating

³ In particular, when $d = 2$ this method also works at finite temperature. In Appendix B.1 we also explicitly extend this method to a covariant version in three dimensions and reproduce the HRT formula.

the entanglement entropy of a subsystem in WCFT to the problem of calculating the thermal entropy. Thirdly, following the logic of deriving the Cardy formula in a CFT [13], [6] calculated the thermal entropy for WCFT by using properties of modular transformations. The idea is to find a modular transformation which exchanges the thermal circle and the spatial circle, and consequently maps the density of high energy states to ground state degeneracy. Note that WCFT is not modular invariant, and hence we need to keep track of the anomalies. Putting all together, [32] managed to write down a formula of entanglement entropy of an arbitrary interval in WCFT.

In this paper, we use a more general set of warped conformal mappings, with one additional parameter α as compared to [32]. On the WCFT side, we recover the results of [32] if we choose $\alpha = 0$. However, as will be seen later, to match the gravity results, we have to choose $\alpha = \pi$ instead.

We begin with the $\text{WCFT}_{(X,T)}$, and consider the following interval

$$\mathcal{A}: \quad \{(X, T) | X = l_X(-\frac{1}{2} + \tau), \quad T = l_T(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}. \quad (3.1)$$

To calculate the entanglement entropy of this interval, the key is to find a suitable warped conformal mapping to a plane with a thermal identification, i.e. the analog of Rindler/Hyperbolic space. Here we would like to apply the following warped conformal mapping which satisfies (2.1)⁴,

$$\frac{\tanh \frac{\pi X}{\bar{\beta}}}{\tanh \frac{l_X \pi}{2\bar{\beta}}} = \tanh \frac{\pi x}{\kappa}, \quad T + \left(\frac{\bar{\beta}}{\beta} - \frac{\alpha}{\beta} \right) X = t + \left(\frac{\bar{\kappa}}{\kappa} - \frac{\alpha}{\kappa} \right) x. \quad (3.2)$$

Let us denote the (x, t) space by \mathcal{H} . The mapping (3.2) ensures that the new theory in \mathcal{H} is also a WCFT, hereafter denoted by $\text{WCFT}_{(x,t)}$. This warped conformal mapping have two key features. Firstly, the (x, t) plane covers a strip region with $-\frac{l_X}{2} < X < \frac{l_X}{2}$, i.e. the shaded region in Fig. 1. Secondly, the mapping induces a thermal circle in \mathcal{H} :

$$thermal: \quad (x, t) \sim (x + i\kappa, t - i\bar{\kappa}). \quad (3.3)$$

As was argued by [32], the entanglement entropy of the interval \mathcal{A} equals to the thermal entropy measured by the ‘‘Rindler observer’’ in \mathcal{H} , thus

$$S_{EE} = -\text{tr}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}) = S_{thermal}(\mathcal{H}). \quad (3.4)$$

The divergence of S_{EE} , which arises from the short distance entanglement near the end points of the interval \mathcal{A} is now mapped to the divergence of the size of the thermal system \mathcal{H} . To see this explicitly we introduce a cutoff ϵ and define a regulated interval

$$\mathcal{A}: \quad \{(X, T) | X = (l_X - 2\epsilon)(-\frac{1}{2} + \tau), \quad T = (l_T - 2\frac{l_T}{l_X}\epsilon)(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}. \quad (3.5)$$

⁴The warped conformal mapping chosen in [32] is (3.2) with the choice $\alpha = 0$. With this difference, we will see that our result of entanglement entropy will take a slightly different form from that of [32].

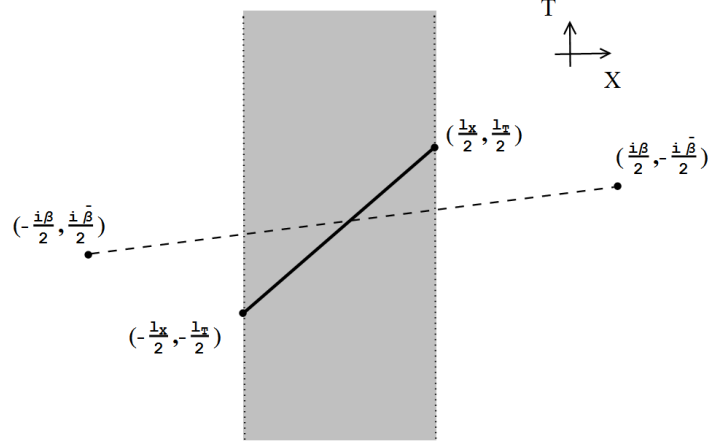


Figure 1. Diagram that depict the region of (X, T) covered by the (x, t) space, which is the shaded strip. The solid line segment is the interval (3.1), and the identification of the dashed line gives the thermal circle.

Notice that the factor in front of the cutoff in the T direction is chosen to guarantee that the regulated interval is contained in the original interval. Using the mapping (3.2), the image of the regulated interval in (x, t) coordinates is given by

$$\{(x, t) | x = 2\pi a(-\frac{1}{2} + \tau), \quad T = 2\pi \bar{a}(\frac{1}{2} - \tau), \quad \tau \in [0, 1]\}, \quad (3.6)$$

$$2\pi a = \frac{\kappa \zeta}{\pi}, \quad 2\pi \bar{a} = \frac{\bar{\kappa} - \alpha}{\pi} \zeta - l_T - l_X(\frac{\bar{\beta} - \alpha}{\beta}), \quad (3.7)$$

where

$$\zeta = \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{l_X \pi}{\beta} \right) + \mathcal{O}(\epsilon). \quad (3.8)$$

Here we have taken the small ϵ limit and only kept the leading term in the expansion. Under this limit, ζ becomes very large.

The image of the regulated interval in the (x, t) coordinates is of infinite length, hence we expect the edge effects can be omitted. Therefore we can identify its endpoints, and denote this identification as the spatial circle

$$\text{spatial circle} : (x, t) \sim (x + 2\pi a, t - 2\pi \bar{a}). \quad (3.9)$$

This circle together with the thermal circle (3.3) form a torus, and the partition function calculated on this torus can be denoted as $Z_{\bar{a}|a}(\bar{\kappa}|\kappa)$. It is useful to perform a further warped conformal mapping

$$\hat{x} = \frac{x}{a}, \quad \hat{t} = t + \frac{\bar{a}}{a}x, \quad (3.10)$$

which changes an arbitrary torus to a canonical torus with a canonical (spatial) circle with $(a, \bar{a}) = (1, 0)$, and a thermal circle independent of the parameters $\kappa, \bar{\kappa}$:

$$(\hat{x}, \hat{t}) \sim (\hat{x} + 2\pi, \hat{t}) \sim (\hat{x} + i\hat{\kappa}, \hat{t} - i\bar{\kappa}), \quad (3.11)$$

with

$$\hat{\kappa} = \frac{\kappa}{a} = \frac{2\pi^2}{\zeta}, \quad \hat{\bar{\kappa}} = \bar{\kappa} - \frac{\bar{a}}{a}\kappa = \alpha + \pi \frac{l_T + \frac{\bar{\beta}-\alpha}{\beta}l_X}{\zeta}. \quad (3.12)$$

To calculate the partition function, one needs to perform a modular transformation S , which exchanges the spatial and thermal circles. The partition function will acquire some additional factors due to the anomaly. The exception is the partition function on a so-called canonical (spatial) circle, $Z_{1,0}(\hat{\bar{\kappa}}|\hat{\kappa}) = Z_{\bar{\kappa}|\kappa}(0|-1)$. Then the high temperature limit on the left hand side becomes a low temperature limit on the right hand side, and the dominate contribution to the later is given by the vacuum expectation values of the generators. By keeping track of the appropriate anomalies (for details see [6, 32]) and taking the limit $\zeta \rightarrow \infty$, we get the dominant contribution

$$Z_{1,0}(\hat{\bar{\kappa}}|\hat{\kappa}) = e^{\frac{k}{4}\frac{\bar{\kappa}^2}{\bar{\kappa}}} Z_{1,0}\left(\frac{2\pi i \hat{\bar{\kappa}}}{\hat{\kappa}} \middle| \frac{4\pi^2}{\hat{\kappa}}\right) = e^{\frac{k}{4}\frac{\bar{\kappa}^2}{\bar{\kappa}}} e^{\frac{2\pi i \bar{\kappa}}{\bar{\kappa}} P_0^{vac} - \frac{4\pi^2}{\bar{\kappa}} L_0^{vac}}, \quad (3.13)$$

where P_0^{vac} and L_0^{vac} are the expectation values of the charges under the limit $\zeta \rightarrow \infty$. (3.13) is valid if the spectrum of P_0 is real or starts from a purely imaginary value with $iP_0^{vac} > 0$, as we will encounter from the bulk analysis. Then the thermal entropy is

$$S_{thermal} = (1 - \hat{\kappa}\partial_{\hat{\kappa}} - \hat{\bar{\kappa}}\partial_{\hat{\bar{\kappa}}}) \log Z_{1,0}(\hat{\bar{\kappa}}|\hat{\kappa}). \quad (3.14)$$

One can check that the entropy is invariant under (3.10), hence (3.14) also gives the thermal entropy of $\text{WCFT}_{(x,t)}$ defined on the original torus, and furthermore gives the entanglement entropy of the interval (3.1). Plugging (3.13) into (3.14) we get the entanglement entropy

$$\begin{aligned} S_{EE}(\mathcal{A}) &= \frac{2\pi(i\hat{\bar{\kappa}}P_0^{vac} - 4\pi L_0^{vac})}{\hat{\kappa}} \\ &= iP_0^{vac}\left(l_T + \frac{\bar{\beta}-\alpha}{\beta}l_X\right) + (i\frac{\alpha}{\pi}P_0^{vac} - 4L_0^{vac}) \log\left(\frac{\beta}{\pi\epsilon} \sinh \frac{l_X\pi}{\beta}\right). \end{aligned} \quad (3.15)$$

Note that when $\zeta \rightarrow \infty$, we have $\frac{4\pi^2}{\bar{\kappa}} \rightarrow \infty$ and $\frac{2\pi i \bar{\kappa}}{\bar{\kappa}} \rightarrow i\infty$ for finite α . The thermal circle is of infinite length, but the angle between the thermal circle and the spatial circle is α dependent. Only when $\alpha = 0$, the thermal circle and the spatial circle become perpendicular. Since P_0^{vac} and L_0^{vac} are the expectation values of the charges under this limit, they will be α dependent, in general. We assume, but do not provide a proof, that the explicit α in (3.15) will be cancelled by the implicit dependence of P_0^{vac} and L_0^{vac} on α . This assumption will be consistent with the assumption that entropy should be invariant under the warped conformal transformations, and will also be consistent with our bulk calculations later. When $\alpha = 0$, (3.15) has the same expression of [32].

For later convenience, we also write down explicitly the combination of the coordinate transformations (3.2) and (3.10)

$$\frac{\tanh \frac{\pi X}{\beta}}{\tanh \frac{l_X\pi}{2\beta}} = \tanh \frac{\pi \hat{x}}{\hat{\kappa}}, \quad T + \left(\frac{\bar{\beta}}{\beta} - \frac{\alpha}{\beta}\right) X = \hat{t} + \left(\frac{\hat{\bar{\kappa}}}{\hat{\kappa}} - \frac{\alpha}{\hat{\kappa}}\right) \hat{x}, \quad (3.16)$$

which is also a warped conformal mapping satisfying (2.1). We denote the WCFT on the (\hat{x}, \hat{t}) as $\text{WCFT}_{(\hat{x}, \hat{t})}$.

4 The gravity side story

In this section, we apply the Rindler method [25] to the gravity side, in the context of AdS/WCFT or WAdS/WCFT. Following the logic of [25], we first find the analog of hyperbolic black holes in the bulk, and then calculate the regularized entropy. We propose that this regularized thermal entropy is just the holographic entanglement entropy for a WCFT. We leave the precise matching between the bulk and boundary calculations to next section.

The simplest model is AdS₃ in Einstein gravity, with the CSS boundary conditions [18]. We will also consider the class of three dimensional theories of gravity obtained from consistent truncations of string theory discussed in [22], [11]. Both BTZ black strings and WAdS₃ black strings are solutions to these theories. An interesting feature is that the thermodynamical properties and the asymptotic symmetries of all the WAdS₃ black strings are identical to those of BTZ after some appropriate reparametrization. In particular, the entropy of all these black strings are given by the horizon length. In this section we will not specify what theory we are using. The discussions below apply to all the models in [22], as well as BTZ black holes and black strings in Einstein gravity. We will only explicitly write down the formulas for WAdS₃, but please keep in mind that it also works for AdS₃ with CSS boundary conditions by setting $\lambda = 0$.

We start with the example with $T_V = 1$, $T_U = 0$ in section 4.1, and in section 4.2 we write down the results for general temperatures.

4.1 The story with $T_V = 1$, $T_U = 0$

In this subsection, we will elaborate the ideas for a simple example of WAdS black string with $T_V = 1$, $T_U = 0$. We first show how to find the coordinate transformation by the quotient method in section 4.1.1, then calculate the thermal entropy of the resulting black hole in section 4.1.2, and finally discuss the geometric quantity that captures the holographic entanglement entropy in section 4.1.3.

4.1.1 The quotient: from $\text{WAdS}_{(U,V,\rho)}$ to $\text{WAdS}_{(u,v,r)}$

To find the analog of hyperbolic black holes, the key is to find a bulk extension of the warped conformal transformation (3.2), which maps one foliation of locally WAdS₃ (AdS₃) space to another foliation. We denote the former spacetime by $\text{WAdS}_{(U,V,\rho)}$, and the later by $\text{WAdS}_{(u,v,r)}$, where the subscripts are the coordinate systems. Analogous to hyperbolic black holes, $\text{WAdS}_{(u,v,r)}$ has to be thermal, with an infinite entropy to potentially reproduce the thermal entropy in $\text{WCFT}_{(x,t)}$. The natural candidate is a WAdS₃ (AdS₃) black string. We will find $\text{WAdS}_{(u,v,r)}$ by doing a $SL(2, R) \times U(1)$ quotient. Note that WAdS₃ has locally well defined isometry $SL(2, R) \times U(1)$, let us denote the generators of $SL(2, R)$ by $J_{0,\pm}$, and the generator of $U(1)$ by J_L . We expect the warped hyperbolic black hole to have two explicit commuting Killing vectors. One Killing vector must be proportional to J_L , while the other must be a linear combination of the four generators. We could further require that the new radial direction is orthogonal to the two Killing vectors. Solving all these conditions we will get the coordinate transformation, as well as the new metric, up to a reparametrization of r , and some integration constants. This method was used to build

and classify locally AdS₃ solutions [45], and was generalized in [5] to find WAdS₃ black hole solutions.

We start with a warped black string WAdS_(U,V,ρ) with $T_V = 1, T_U = 0$

$$ds^2 = \ell^2 \left(\frac{(\lambda^2 + 1) d\rho^2}{4\rho^2} - \lambda^2 \rho^2 dU^2 + 2\rho dU dV + dV^2 \right), \quad (4.1)$$

with Killing vectors

$$\begin{aligned} J_L &= \partial_V, \\ J_+ &= \frac{4\rho^2 U^2 + 1}{4\rho^2} \partial_U - \frac{1}{2\rho} \partial_V - 2\rho U \partial_\rho, \\ J_0 &= U \partial_U - \rho \partial_\rho, \\ J_- &= \partial_U, \end{aligned} \quad (4.2)$$

where the normalization are chosen to satisfy the standard $SL(2, R)$ algebra $[J_-, J_+] = 2J_0$, $[J_0, J_\pm] = \pm J_\pm$. Define

$$J = a_L J_L + a_0 J_0 + a_+ J_+ + a_- J_-, \quad (4.3)$$

where a_0, a_+, a_-, a_L are arbitrary constants. We define some new coordinates such that there are two explicit Killing vectors

$$\partial_u = J, \quad \partial_v = J_L, \quad (4.4)$$

where we fix the ratio between ∂_v and J_L by requiring the new metric satisfies the same asymptotic boundary conditions with old metric WAdS_(U,V,ρ). From (4.4) we get some components of the new metric

$$g_{uu} = J \cdot J, \quad g_{uv} = J \cdot J_L, \quad g_{vv} = J_L \cdot J_L, \quad (4.5)$$

where the inner products are calculated with the old metric for WAdS_(U,V,ρ). The new metric should only depend on the third coordinate r . Up to reparameterization, we can always choose the new radial coordinates by

$$r \equiv \frac{J \cdot J_L}{\ell^2}. \quad (4.6)$$

It is easy to verify that g_{uu} and g_{vv} only depend on r . We further require that the new metric has no cross terms between r and u, v , namely

$$g_{ru} = n_r \cdot J = 0, \quad g_{rv} = n_r \cdot J_L = 0, \quad n_r \equiv \partial_r. \quad (4.7)$$

Solving all these conditions will give the coordinate transformation, as well as the new metric.

The most general quotient with arbitrary parameters a_0, a_+, a_-, a_L is given in Appendix A. The boundary coordinates (u, v) covers a strip of (U, V) , with $a_0/(2a_+)$ controlling the center position of strip, while $-\sqrt{a_0^2 - 4a_- a_+}/a_+$ controlling the width of the strip. It

turns out that the regularized Bekenstein-Hawking entropy of the $\text{WAdS}_{(u,v,r)}$ only depend on the width of the strip. Hence, for simplicity but without losing generality, we choose the parameters in the main text as follows

$$a_L = a_0 = 0, \quad a_+ = -\frac{2}{l_U}, \quad a_- = \frac{l_U}{2}. \quad (4.8)$$

With the above choice of the parameters, we find the following coordinate transformation

$$\begin{aligned} u &= \frac{1}{4} \log \left(\frac{(l_U + 2U)^2 \rho^2 - 1}{(l_U - 2U)^2 \rho^2 - 1} \right), \\ v &= \frac{1}{4} \log \left(\frac{(1 + 2\rho U)^2 - l_U^2 \rho^2}{(1 - 2\rho U)^2 - l_U^2 \rho^2} \right) + V, \\ r &= \frac{1 + \rho^2 (l_U^2 - 4U^2)}{2l_U \rho}, \end{aligned} \quad (4.9)$$

Under which we get a new warped black string, denoted by $\text{WAdS}_{(u,v,r)}$,

$$ds^2 = \ell^2 \left((T_u^2 (1 + \lambda^2 T_v^2) - \lambda^2 r^2) du^2 + 2r dudv + T_v^2 dv^2 + \frac{(1 + \lambda^2 T_v^2)}{4(r^2 - T_u^2 T_v^2)} dr^2 \right), \quad (4.10)$$

$$T_u = T_v = 1, \quad (4.11)$$

with an infinite event horizon at

$$r_h = T_u T_v. \quad (4.12)$$

It is easy to verify that $\text{WAdS}_{(u,v,r)}$ (4.10) and $\text{WAdS}_{(U,V,\rho)}$ (4.1) satisfy the same Dirichlet-Neumann type of boundary conditions.

4.1.2 A bulk calculation of the entanglement entropy

Similar to the story of $\text{AdS}_3/\text{CFT}_2$, the $\text{WAdS}_3/\text{WCFT}$ dictionary will translate the bulk calculation to a boundary calculation. As discussed in section 2, asymptotic symmetry analysis directly relate gravity on $\text{WAdS}_{(U,V,\rho)}$ to $\text{WCFT}_{(U,V)}$ with the tilded algebra (2.6).

Let us look at the coordinate transformations on the boundary, which are given by

$$\begin{aligned} u &= \frac{1}{2} \log \left(\frac{l_U + 2U}{l_U - 2U} \right) + \mathcal{O}\left(\frac{1}{\rho^2}\right), \\ v &= V + \mathcal{O}\left(\frac{1}{\rho}\right). \end{aligned} \quad (4.13)$$

The boundary coordinate transformation (4.13) indicates that the boundary of $\text{WAdS}_{(u,v,r)}$ covers a strip with $-\frac{l_U}{2} < U < \frac{l_U}{2}$, on the boundary of $\text{WAdS}_{(U,V,\rho)}$.

Similar to the discussion on the field theory side, now let us consider an interval

$$\mathcal{A}: \{(U, V) | U = (l_U - 2\epsilon)(-\frac{1}{2} + \tau), \quad V = l_V(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}, \quad (4.14)$$

with ϵ being a small cutoff. We find that on the boundary of $\text{WAdS}_{(u,v,r)}$, the interval (4.14) in terms of the new coordinates (u, v) is given by

$$\{(u, v) | u = \Delta u(-\frac{1}{2} + \tau), \quad v = \Delta v(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}, \quad (4.15)$$

with

$$\Delta u = \log \left(\frac{l_U}{\epsilon} \right), \quad \Delta v = l_V. \quad (4.16)$$

We see the interval (4.15) is infinitely extended in the u direction as $\epsilon \rightarrow 0$. As in section 2.2.3, we identify the end points of the interval. Thus, beside the thermal circle, we also have a spatial circle in $\text{WAdS}_{(u,v,r)}$

$$\text{spatial circle} : (u, v) \sim (u + \Delta u, v + \Delta v), \quad (4.17)$$

According to our discussions in section 2, given the temperatures $T_u = T_v = 1$, the total thermal entropy of $\text{WAdS}_{(u,v,r)}$ (4.10) on the interval (4.15) is given by (2.24). In terms of variables of $\text{WAdS}_{(U,V,\rho)}$ and using (4.16), we get

$$S_{HEE} = \frac{\ell}{4G} l_V + \frac{\ell}{4G} \log \frac{l_U}{\epsilon}. \quad (4.18)$$

We propose that (4.18) is the bulk calculation for the entanglement entropy in the context of WAdS/WCFT and AdS/WCFT . We will show explicitly how (4.18) reproduce the WCFT result (3.15) in next section.

4.1.3 The geometric quantity in $\text{WAdS}_{(U,V,\rho)}$

We can also calculate the inverse coordinate transformations from $\text{WAdS}_{(u,v,r)}$ to $\text{WAdS}_{(U,V,\rho)}$, and find out what is the image of the horizon of $\text{WAdS}_{(u,v,r)}$. The inverse coordinate transformations are ⁵,

$$U = \frac{l_U \sqrt{r^2 - 1} \sinh(2u)}{2 \left(\sqrt{r^2 - 1} \cosh(2u) + r \right)}, \quad (4.20)$$

$$\rho = \frac{\left(r + \sqrt{r^2 - 1} \cosh(2u) \right)}{l_U}, \quad (4.21)$$

$$V = \frac{1}{4} \log \left(\frac{2\sqrt{r^2 - 1}e^{2u} + (r + 1)e^{4u} + r - 1}{2\sqrt{r^2 - 1}e^{2u} + (r - 1)e^{4u} + r + 1} \right) + v. \quad (4.22)$$

We parametrize the horizon of $\text{WAdS}_{(u,v,r)}$ in the following way

$$\{(u, v, r) | u = -\frac{\Delta u}{2} + \tau \Delta u, \quad v = -\frac{\Delta v}{2} + \tau \Delta v, \quad r = 1, \quad \tau \in [0, 1]\}. \quad (4.23)$$

⁵There are two branches of the inverse coordinate transformations and both satisfy (4.9). The other branch is given by

$$U = \frac{l_U \sqrt{r^2 - 1} \sinh(2u)}{2 \left(\sqrt{r^2 - 1} \cosh(2u) - r \right)}, \quad \rho = \frac{(r - \sqrt{r^2 - 1} \cosh(2u))}{l_U}. \quad (4.19)$$

We drop this branch since it will give negative ρ when u is big enough.

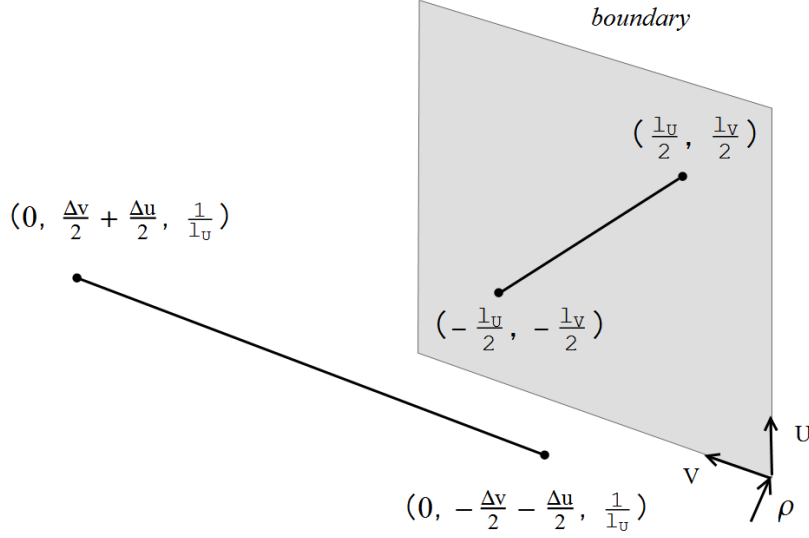


Figure 2. The solid line segment in the bulk is the geometric quantity $\gamma_{\mathcal{A}}$ (4.24) whose length is proportional to the entanglement entropy of the interval on the boundary (4.14).

The inverse coordinate transformations (4.20)-(4.22) indicate that the image interval of the horizon (4.23), denoted by $\gamma_{\mathcal{A}}$, is an interval with both the ρ and U coordinates fixed (See Fig.2), depicted by

$$\gamma_{\mathcal{A}} : \{(U_h, V_h, \rho_h) | U_h = 0, \quad \rho_h = \frac{1}{l_U}, \quad V_h = (\Delta u + \Delta v)(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}. \quad (4.24)$$

Using (4.16) and $g_{VV} = \ell^2$, we find the proposed holographic entanglement entropy (4.18) is given by the length of $\gamma_{\mathcal{A}}$

$$S_{HEE} = \frac{Length(\gamma_{\mathcal{A}})}{4G}, \quad (4.25)$$

However, it is surprising that the interval $\gamma_{\mathcal{A}}$, which plays the role of the minimal surface in the Ryu-Takayanagi formula [23, 24], is not the minimal surface anchored on the end points of the interval (4.14) on the asymptotic boundary. This is the main difference between this bulk calculation in WAdS and the RT proposal for AdS.

4.2 The story with arbitrary temperatures

In this subsection we extend our discussions to WAdS $_{(U,V,\rho)}$ with arbitrary temperatures

$$ds^2 = \ell^2 \left((T_U^2 (\lambda^2 T_V^2 + 1) - \lambda^2 \rho^2) dU^2 + 2\rho dU dV + T_V^2 dV^2 + \frac{(\lambda^2 T_V^2 + 1) d\rho^2}{4(\rho^2 - T_U^2 T_V^2)} \right). \quad (4.26)$$

Similar to the case of $T_V = 1, T_U = 0$, we can directly use the quotient method following the steps in section 4.1. Alternatively we can first perform a coordinate transformation to reduce the metric (4.26) of WAdS $_{(U,V,\rho)}$ to the metric (4.1) with $T_U = 0, T_V = 1$, and

then use the result in section 4.1. This amounts to performing the following coordinate transformation

$$\begin{aligned}
4T_U U &= \log \left(\frac{\left(\sqrt{r^2 - T_V^2} \left(\cosh(2u) + \sinh(2u) \tanh\left(\frac{l_U T_U}{2}\right) \right) + r \right)^2 - T_V^2 \tanh^2\left(\frac{l_U T_U}{2}\right)}{\left(\sqrt{r^2 - T_V^2} \left(\cosh(2u) - \sinh(2u) \tanh\left(\frac{l_U T_U}{2}\right) \right) + r \right)^2 - T_V^2 \tanh^2\left(\frac{l_U T_U}{2}\right)} \right), \\
2T_V V &= 2T_V v + \coth^{-1} \left(\frac{\operatorname{csch}(2u)}{T_V} \left(\sqrt{r^2 - T_V^2} + r \cosh(2u) \right) \right) - \\
&\quad \tanh^{-1} \left(\frac{4T_V \sqrt{r^2 - T_V^2} \sinh(2u) \tanh^2\left(\frac{l_U T_U}{2}\right)}{\operatorname{sech}^2\left(\frac{l_U T_U}{2}\right) \left((r^2 - T_V^2) \cosh(4u) - r^2 + 3T_V^2 \right) + 4r \sqrt{r^2 - T_V^2} \cosh(2u) + 4(r^2 - T_V^2)} \right), \\
\rho &= T_U \operatorname{csch}(l_U T_U) \left(r \cosh(l_U T_U) + \sqrt{r^2 - T_V^2} \cosh(2u) \right).
\end{aligned} \tag{4.27}$$

Under (4.27) we get the metric of $\text{WAdS}_{(u,v,r)}$

$$ds^2 = (T_u^2 (1 + \lambda^2 T_v^2) - \lambda^2 r^2) du^2 + 2r dudv + T_v^2 dv^2 + \frac{(1 + \lambda^2 T_v^2)}{4(r^2 - T_u^2 T_v^2)} dr^2, \tag{4.28}$$

$$T_u = 1, \quad T_v = T_V. \tag{4.29}$$

The coordinate transformation on the boundary is given by

$$\begin{aligned}
\frac{\tanh(T_U U)}{\tanh\left(\frac{l_U T_U}{2}\right)} &= \tanh(u) + \mathcal{O}\left(\frac{1}{r^2}\right), \\
V &= v + \mathcal{O}\left(\frac{1}{r}\right),
\end{aligned} \tag{4.30}$$

which similarly indicate that the boundary of $\text{WAdS}_{(u,v,r)}$ covers a strip on the boundary $\text{WAdS}_{(U,V,\rho)}$ of with width l_U .

Again we consider an interval (4.14), and find that on the boundary of $\text{WAdS}_{(u,v,r)}$, the interval (4.14) in terms of the new coordinates (u, v) is given by (4.15), but with

$$\Delta u = \log \left(\frac{\sinh(l_U T_U)}{\epsilon T_U} \right), \quad \Delta v = l_V. \tag{4.31}$$

Again this interval (4.15) is infinitely extended along the u direction, and hence we identify the end points, which lead to a spatial circle

$$\text{spatial circle} : (u, v) \sim (u + \Delta u, v + \Delta v), \tag{4.32}$$

According to our discussions in Sec. 2, the thermal entropy of $\text{WAdS}_{(u,v,r)}$ (4.28) with the above spatial circle is given by

$$S_{HEE} = \frac{\ell}{4G} \left(T_V l_V + \log \left(\frac{\sinh(l_U T_U)}{\epsilon T_U} \right) \right). \tag{4.33}$$

We can also find the image interval $\gamma_{\mathcal{A}}$ of the horizon of $\text{WAdS}_{(u,v,r)}$

$$\{(U_h, V_h, \rho_h) | U_h = 0, \rho_h = T_U T_V \coth(l_U T_U), V_h = (\frac{T_V \Delta v + \Delta u}{T_V})(-\frac{1}{2} + \tau), \tau \in [0, 1]\}. \quad (4.34)$$

Using (4.31) and $g_{VV} = \ell^2 T_V^2$, we find the length of $\gamma_{\mathcal{A}}$ is given by

$$\text{Length}(\gamma_{\mathcal{A}}) = \ell \left(T_V l_V + \log \left(\frac{\sinh(l_U T_U)}{\epsilon T_U} \right) \right), \quad (4.35)$$

and also

$$S_{HEE} = \frac{\text{Length}(\gamma_{\mathcal{A}})}{4G}. \quad (4.36)$$

5 Matching between WAdS_3 and the WCFT

In this section, we relate the entanglement entropy in the WCFT (3.15) to the bulk calculation (4.33). Section 5.1 relates the $\text{WCFT}_{(U,V)}$, $\text{WCFT}_{(u,v)}$ and $\text{WCFT}_{(\hat{u},\hat{v})}$ appearing from the bulk analysis to a $\text{WCFT}_{(X,T)}$, $\text{WCFT}_{(x,t)}$ and $\text{WCFT}_{(\hat{x},\hat{t})}$ with canonical algebra. Section 5.2 shows that the bulk coordinate transformation is indeed an extension of the warped conformal transformation. Section 5.3 shows how to determine the vacuum charges using holography. Section 5.4 finally shows that the bulk result and the WCFT result agree with each other.

5.1 $\text{WAdS}_{(U,V,\rho)}$, $\text{WCFT}_{(U,V)}$ and $\text{WCFT}_{(X,T)}$

As discussed in Sec. 2.2, the asymptotic symmetry of AdS_3 and WAdS_3 under the Dirichlet-Neumann boundary conditions is given by the algebra (2.6). Given a $\text{WAdS}_{(U,V,\rho)}$ with arbitrary temperatures T_U and T_V (4.26), the boundary theory $\text{WCFT}_{(U,V)}$ is naturally defined with coordinates U and V . The algebra (2.6) is related to the canonical WCFT algebra (2.2) by a redefinition of the generators. In particular, for states with vanishing $P_{n \neq 0}$, the mapping can be achieved by a (state-dependent) coordinate transformation (2.14),

$$U = X, \quad V = \frac{T}{T_V} + X, \quad (5.1)$$

where we have chosen $k = -\frac{\ell}{G}$, which means that $P_0 = -\sqrt{k\tilde{P}_0} = -\frac{\ell}{2G}T_V$. We will denote the WCFT in the (X, T) coordinates by $\text{WCFT}_{(X,T)}$. The thermal circle of the black string (4.26) is

$$\text{thermal circle : } (U, V) \sim (U + \frac{\pi i}{T_U}, V - \frac{\pi i}{T_V}), \quad (5.2)$$

Under the coordinate transformation (5.1), the above thermal circle (5.2) induces a thermal circle for $\text{WCFT}_{(X,T)}$

$$\text{thermal circle : } (X, T) \sim (X + i\beta, T - i\bar{\beta}), \quad (5.3)$$

$$\beta = \frac{\pi}{T_U}, \quad \bar{\beta} = \pi \left(\frac{T_V}{T_U} + 1 \right). \quad (5.4)$$

Similarly, a spatial interval on the (U, V) plane parameterized by l_U, l_V will be mapped to a spatial interval with

$$\text{spatial interval : } \{(X, T) | X = l_X(-\frac{1}{2} + \tau), \quad T = l_T(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}, \quad (5.5)$$

$$l_T = T_V(l_V - l_U), \quad l_X = l_U. \quad (5.6)$$

According to the discussions in section 3, the entanglement entropy for a spatial interval (5.5) on the $\text{WCFT}_{(X,T)}$ with a thermal circle (5.3) is given by (3.15).

5.2 $\text{WAdS}_{(u,v,r)}$, $\text{WCFT}_{(u,v)}$ and $\text{WCFT}_{(x,t)}$

In this subsection we map $\text{WCFT}_{(u,v)}$, obtained from $\text{WAdS}_{(u,v,r)}$, to $\text{WCFT}_{(x,t)}$, which will relate the bulk coordinate transformation (4.27) and the warped conformal transformation (3.2).

Similar to the previous subsection, we get $\text{WCFT}_{(u,v)}$ from asymptotic analysis of $\text{WAdS}_{(u,v,r)}$. The transformation

$$u = x, \quad v = \frac{t}{T_v} + x, \quad (5.7)$$

leads to a torus

$$\begin{aligned} \text{spatial} : (x, t) &\sim (x + \Delta u, t + T_v(\Delta v - \Delta u)), \\ \text{thermal} : (x, t) &\sim (x + i\frac{\pi}{T_u}, t - iT_v\left(\frac{\pi}{T_u} + \frac{\pi}{T_v}\right)). \end{aligned} \quad (5.8)$$

Note that the bulk coordinate transformation (4.27) induces a warped conformal mapping from $\text{WCFT}_{(U,V)}$ to $\text{WCFT}_{(u,v)}$, given by (4.30). (4.30) can be rewritten in terms of variables in the (X, T) and (x, t) coordinates using (5.1) and (5.7)

$$\frac{\tanh \frac{\pi X}{\beta}}{\tanh \frac{l_X \pi}{2\beta}} = \tanh x, \quad T + T_V X = t + T_v x. \quad (5.9)$$

Comparing to the warped conformal transformation (3.2), it is easy to read the three parameters

$$\kappa = \frac{\pi}{T_u}, \quad \bar{\kappa} = T_v \left(\frac{\pi}{T_u} + \frac{\pi}{T_v} \right), \quad (5.10)$$

$$\alpha = \pi. \quad (5.11)$$

where we have used $T_u = 1, T_v = T_V$. Note that by choosing arbitrary $a_+ a_-$ and a_L in (4.3), we can also get arbitrary T_u and T_v . This is consistent with the fact κ and $\bar{\kappa}$ are arbitrary and will not affect the entanglement entropy. On the other hand, α shows up explicitly in the entanglement entropy, and (5.11) is the matching condition between the bulk and the boundary. As a consistent check, we can rewrite the torus (5.8) in terms of the x, t variables by using (4.31), (5.4) and (5.6), it is straight forward to see that indeed (5.8) agrees with (3.9) and (3.3). Thus we proved that the bulk coordinate transformations (4.27) is indeed a bulk extension of the warped conformal mapping (3.2) with the choice (5.10) and (5.11) on the field theory side.

5.3 WAdS_($\hat{u}, \hat{v}, \hat{r}$), WCFT_(\hat{u}, \hat{v}) and WCFT_(\hat{x}, \hat{t})

In this subsection, we map WCFT_(\hat{u}, \hat{v}) to WCFT_(\hat{x}, \hat{t}). This will fix P_0^{vac} and L_0^{vac} in the formula of entanglement entropy (3.15). Meanwhile we will further check the consistency of all transformations.

As discussed in section 2.2.3, we apply the following transformation to the WAdS_(u, v, r) metric (4.28),

$$u = \frac{\Delta u}{2\pi} \hat{u}, \quad v = \frac{\Delta v}{2\pi} \hat{v}, \quad r = \frac{4\pi^2}{\Delta v \Delta u} \hat{r}, \quad (5.12)$$

$$\frac{\Delta u}{2\pi} = T_{\hat{u}}, \quad \frac{\Delta v}{2\pi} T_v = T_{\hat{v}}, \quad \lambda \frac{2\pi}{\Delta v} = \hat{\lambda}. \quad (5.13)$$

and get WAdS_($\hat{u}, \hat{v}, \hat{r}$)

$$ds^2 = \left(T_{\hat{u}}^2 \left(1 + \hat{\lambda}^2 T_{\hat{v}}^2 \right) - \hat{\lambda}^2 \hat{r}^2 \right) d\hat{u}^2 + 2\hat{r} d\hat{u} d\hat{v} + T_{\hat{v}}^2 d\hat{v}^2 + \frac{\left(1 + \hat{\lambda}^2 T_{\hat{v}}^2 \right)}{4(\hat{r}^2 - T_{\hat{u}}^2 T_{\hat{v}}^2)} d\hat{r}^2, \quad (5.14)$$

with a spatial circle

$$spatial : (\hat{u}, \hat{v}) \sim (\hat{u} + 2\pi, \hat{v} + 2\pi). \quad (5.15)$$

With the Dirichlet-Newmann type of boundary conditions, the holographic dual is WCFT_(\hat{u}, \hat{v}) we discussed in section (2.2.1). Following section (2.2.2), a state dependent transformation with $k = -\frac{\ell}{G}$

$$\hat{u} = \hat{x}, \quad \hat{v} = \frac{\hat{t}}{T_{\hat{v}}} + \hat{x}, \quad (5.16)$$

maps WCFT_(\hat{u}, \hat{v}) to WCFT_(\hat{x}, \hat{t}).

Now we double check that the connection between the bulk and boundary transformations. From (5.16), we get WCFT_(\hat{x}, \hat{t}) on the canonical torus

$$spatial : (\hat{x}, \hat{t}) \sim (\hat{x} + 2\pi, \hat{t}),$$

$$thermal : (\hat{x}, \hat{t}) \sim \left(\hat{x} + i \frac{2\pi^2}{\zeta}, \hat{t} - i\pi \left(1 + \frac{l_T + \frac{\bar{\beta} - \pi}{\beta} l_X}{\zeta} \right) \right), \quad (5.17)$$

where (4.31) and (5.6) are used. Comparing (5.17) with (3.12), we again get a matching with the condition (5.11).

The boundary coordinate transformation from (U, V) to (\hat{u}, \hat{v}) is given by

$$\frac{\tanh(T_U U)}{\tanh\left(\frac{l_U T_U}{2}\right)} = \tanh\left(\frac{\Delta u}{2\pi} \hat{u}\right), \quad V = \frac{\Delta v}{2\pi} \hat{v}. \quad (5.18)$$

Using (5.1), (5.16), (4.31), (5.4) and (5.6), we get a transformation between X, T and (\hat{x}, \hat{t})

$$\frac{\tanh \frac{\pi X}{\beta}}{\tanh \frac{l_X \pi}{2\beta}} = \tanh \frac{\pi \hat{x}}{\hat{\kappa}}, \quad T + \left(\frac{\bar{\beta}}{\beta} - \frac{\pi}{\beta} \right) X = \hat{t} + \left(\frac{\hat{\kappa}}{\kappa} - \frac{\pi}{\kappa} \right) \hat{x}. \quad (5.19)$$

which is just the warped conformal mapping (3.16) with $\alpha = \pi$. Thus we again proved that the bulk coordinate transformations (4.27) is indeed a bulk extension of the warped conformal mapping (3.16) with $\alpha = \pi$ on the field theory side.

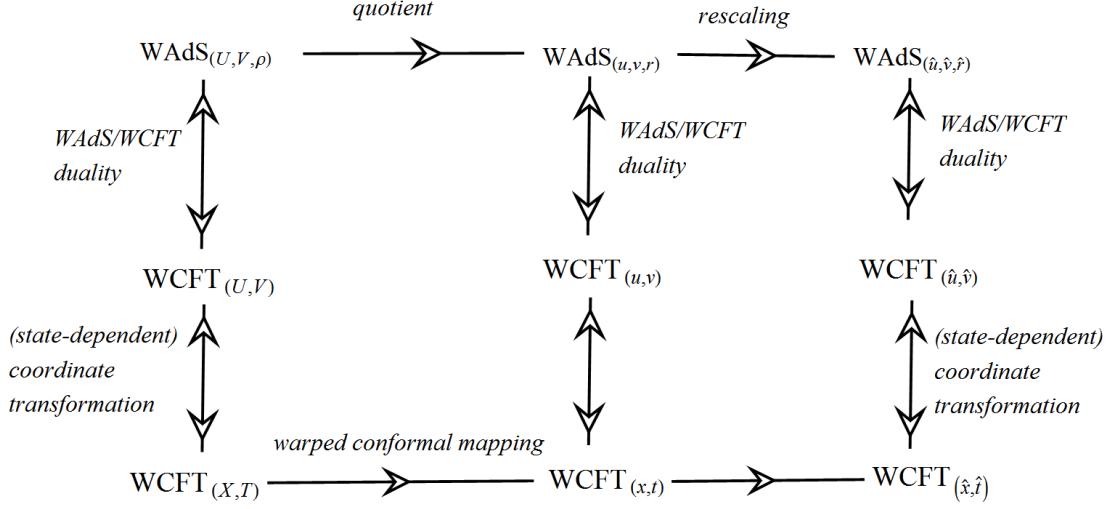


Figure 3. Diagram that make a conclusion about all the spacetimes and field theories we have discussed, and their relationships.

5.4 Matching the entanglement entropy

Now we recite results from both sides, and show the matching. On the gravity side, the proposed holographic entanglement entropy is given by (4.33)

$$S_{HEE} = \frac{\ell}{4G} T_V l_V + \frac{\ell}{4G} \log \left(\frac{\sinh(l_U T_U)}{\epsilon T_U} \right), \quad (5.20)$$

while the entanglement entropy from WCFT analysis is given by (3.15),

$$S_{EE} = i P_0^{vac} (l_T + \frac{\bar{\beta} - \alpha}{\beta} l_X) + (i \frac{\alpha}{\pi} P_0^{vac} - 4 L_0^{vac}) \log \left(\frac{\beta}{\pi \epsilon} \sinh \frac{l_X \pi}{\beta} \right). \quad (5.21)$$

Using the vacuum values of the charges (2.18), which we rewrite here

$$P_0^{vac} = -\frac{i\ell}{4G}, \quad L_0^{vac} = 0, \quad (5.22)$$

and the matching conditions (5.4), (5.6), (5.11), it is straightforward to verify that indeed the bulk result agrees with the boundary result (3.15)

$$S_{HEE} = S_{EE}. \quad (5.23)$$

To recapitulate, our main steps are depict in Fig.3, where the top line shows the calculation on the gravity side and the bottom line shows the calculation on the field theory side. The vertical arrows show how to relate the bulk calculation to the field theory calculation.

6 Rényi entropy

6.1 Rényi entropies for WCFT

The Rényi entropies is quite straight forward to calculate after we have analysed the partition functions. It can be calculated by

$$S_n = \frac{1}{1-n} \log \left(\frac{Z_{a|\bar{a}}(n\bar{\theta}|n\theta)}{Z_{a|\bar{a}}(\bar{\theta}|\theta)^n} \right). \quad (6.1)$$

Using (3.13) and the transformation rules of the partition function

$$Z_{a,\bar{a}}(\bar{\kappa}|\kappa) = e^{\frac{\kappa\bar{a}}{2}(\bar{\kappa} - \frac{\kappa\bar{a}}{2a})} Z_{1,0} \left(\bar{\kappa} - \frac{\bar{a}\kappa}{a} \middle| \frac{\kappa}{a} \right), \quad (6.2)$$

we find

$$S_n = iP_0^{vac}(l_T + \frac{\bar{\beta} - \alpha}{\beta} l_X) + \left(i\frac{\alpha}{\pi} P_0^{vac} - \frac{2(n+1)L_0^{vac}}{n} \right) \log \left(\frac{\beta}{\pi\epsilon} \sinh \frac{l_X \pi}{\beta} \right). \quad (6.3)$$

6.2 Partition function and Rényi entropy calculated on the gravity side

Following the spirit of [44], in this subsection we calculate the Rényi entropy holographically on the gravity side. As we have mentioned previously, the warping parameter does not affect the thermal properties of these WAdS spacetimes, for simplicity we set it to be zero and consider the warped black string in the (\hat{x}, \hat{t}) coordinates

$$ds^2 = \ell^2 \left(d\hat{t}^2 + 2\frac{r + T_{\hat{v}}^2}{T_{\hat{v}}} d\hat{t}d\hat{x} + (2r + T_{\hat{v}}^2 + T_{\hat{u}}^2) d\hat{x}^2 + \frac{dr^2}{4(r^2 - T_{\hat{v}}^2 T_{\hat{u}}^2)} \right). \quad (6.4)$$

This metric is just WAdS $_{(\hat{u}, \hat{v}, \hat{r})}$ (5.14) under the transformation (5.16). The variation of conserved charges δQ_ξ associate with the Killing vector ξ^μ between two background with metric $g + \delta g$ and g is given by,

$$\delta Q_\xi = \int_{\partial\Sigma} \epsilon_{\mu\nu\rho} K^{\mu\nu} (\delta g, g) dx^\rho. \quad (6.5)$$

For example, for Einstein gravity with a negative cosmological constant in 3 dimensions

$$S_{3d} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left(R - \frac{2}{\ell^2} \right), \quad (6.6)$$

$K_\xi^{\mu\nu}$ has the only contribution from the metric,

$$K_\xi^{\mu\nu} = \frac{1}{16\pi G} \left(\xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\rho\nu} \nabla_\rho \xi^\mu \right), \quad (6.7)$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$, $h^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma}$, $h = g^{\mu\nu} h_{\mu\nu}$, and ∇_μ is the covariant derivative compatible with $g_{\mu\nu}$. Now, it is straightforward to calculate the variation of E and P , which are charges associated with $\partial/\partial\hat{t}$ and $\partial/\partial\hat{x}$, and after a trivially integration we get

$$E = \frac{\ell}{2G} T_{\hat{v}}, \quad P = \frac{\ell}{4G} (T_{\hat{v}}^2 - T_{\hat{u}}^2). \quad (6.8)$$

The generator of the horizon is taken to be,

$$\xi_H = \partial_{\hat{t}} + \Omega_H \partial_{\hat{x}}, \quad (6.9)$$

which indicates the angular potential Ω_H is given by

$$\Omega_H = -\frac{1}{T_{\hat{v}} + T_{\hat{u}}}. \quad (6.10)$$

Also we can calculate the Hawking temperature and Bekenstein-Hawking entropy

$$T_H = \frac{T_{\hat{u}}}{\pi(T_{\hat{v}} + T_{\hat{u}})}, \quad S = \frac{\pi\ell}{2G}(T_{\hat{v}} + T_{\hat{u}}). \quad (6.11)$$

With all these physical quantities calculated, we find that the thermodynamical first law $T_H \delta S = \delta E + \Omega_H \delta P$ is satisfied. The partition function of the gravity theory with angular momentum takes the form

$$Z(T_H, \Omega_H) = \exp\left(-\frac{1}{T_H} \Psi(T_H, \Omega_H)\right), \quad (6.12)$$

where $\Psi(T_H, \Omega_H)$ is the grand potential which is given by

$$\Psi = E - T_H S + \Omega_H P = \frac{\ell}{4G}(T_{\hat{v}} - T_{\hat{u}}). \quad (6.13)$$

Thus the gravity partition function is given by

$$Z(T_{\hat{u}}, T_{\hat{v}}) = \exp\left(\frac{\pi\ell}{4G} \frac{T_{\hat{u}}^2 - T_{\hat{v}}^2}{T_{\hat{u}}}\right). \quad (6.14)$$

The Rényi entropy is defined by

$$S_n = \frac{1}{1-n} \log \text{Tr}(\rho^n), \quad (6.15)$$

where ρ is the normalized density matrix

$$\rho = \frac{\exp\left[-\frac{1}{T_H}(\hat{E} + \Omega_H \hat{P})\right]}{Z(T_H, \Omega_H)}. \quad (6.16)$$

It can be shown that

$$\begin{aligned} S_n^{bk} &= \frac{1}{1-n} \log \left[\frac{Z(T_H/n, \Omega_H)}{Z(T_H, \Omega_H)^n} \right] \\ &= \frac{\pi\ell}{2G}(T_{\hat{v}} + T_{\hat{u}}) \end{aligned} \quad (6.17)$$

$$= S_{HEE} \quad (6.18)$$

where in the last line we have used the transformation between (\hat{u}, \hat{v}) and (U, V) to rewrite S_n^{bk} as the proposed holographic entanglement entropy (4.33).

6.3 Matching

Using the vacuum values of the charges (5.22), we find that the Rényi entropy is independent of n , and is just the same as entanglement entropy

$$S_n^{bn} = S_{EE}. \quad (6.19)$$

As was discussed in the previous section, the bulk and boundary calculations of entanglement entropy agree. Therefore, the bulk and boundary calculations for the Rényi entropy also agree,

$$S_n^{bh} = S_{HEE} = S_{EE} = S_n^{bn}. \quad (6.20)$$

Another consistent check is for the partition functions. The torus of WCFT $_{(\hat{x}, \hat{t})}$ is parameterized by

$$\hat{\kappa} = \pi T_{\hat{v}} \left(\frac{1}{T_{\hat{u}}} + \frac{1}{T_{\hat{v}}} \right), \quad \hat{\kappa} = \frac{\pi}{T_{\hat{u}}}. \quad (6.21)$$

Substituting $P_0^{vac} = -\frac{i\ell}{4G}$, $L_0^{vac} = 0$, $k = -\frac{\ell}{G}$ and Eq. (6.21) into (3.13), it is straightforward to show that,

$$Z_{1,0}(\hat{\kappa}|\hat{\kappa}) = \exp \left(\frac{\pi\ell}{4G} \frac{T_{\hat{u}}^2 - T_{\hat{v}}^2}{T_{\hat{u}}} \right), \quad (6.22)$$

which is identical to the gravity partition function (6.14).

Acknowledgement

We thank H. Casini, A. Castro, G. Compère, T. Hartman, D. Hofman, H. Jiang, R. Miao, A. Strominger and J. Wu for helpful discussions. This work was supported in part by start-up funding 543310007 from Tsinghua University. W.S. is also supported by the National Thousand-Young-Talents Program of China.

A The general quotient

Here we give the general quotient of WAdS $_{(U,V,\rho)}$ (4.1) with arbitrary parameters a_0, a_+, a_-, a_L . The corresponding coordinate transformation is given by

$$\begin{aligned} u &= \frac{\coth^{-1} \left(\frac{\sqrt{a_0^2 - 4a_- a_+ \rho}}{a_0 \rho - a_+ (2\rho U + 1)} \right) + \coth^{-1} \left(\frac{\sqrt{a_0^2 - 4a_- a_+ \rho}}{-2a_+ \rho U + a_0 \rho + a_+} \right)}{\sqrt{a_0^2 - 4a_- a_+}}, \\ v &= V + \frac{1}{4} \log \left(\frac{4a_- \rho^2 + (2\rho U + 1)(a_+ (2\rho U + 1) - 2a_0 \rho)}{4a_- \rho^2 + (2\rho U - 1)(a_+ (2\rho U - 1) - 2a_0 \rho)} \right) \\ &\quad + \frac{a_L \left(\coth^{-1} \left(\frac{\sqrt{a_0^2 - 4a_- a_+ \rho}}{2a_+ \rho U - a_0 \rho + a_+} \right) - \coth^{-1} \left(\frac{\sqrt{a_0^2 - 4a_- a_+ \rho}}{-2a_+ \rho U + a_0 \rho + a_+} \right) \right)}{\sqrt{a_0^2 - 4a_- a_+}}, \\ r &= \frac{L^2 (4\rho (a_L + \rho U (a_+ U - a_0) + a_- \rho) - a_+)}{4\rho}. \end{aligned} \quad (A.1)$$

The resulting metric of $\text{WAdS}_{(u,v,r)}$ is

$$ds^2 = \frac{r^2 - (\lambda^2 + 1)(r - r_-)(r - r_+)}{\ell^2} du^2 + \ell^2 dv^2 + 2r dudv + \frac{\ell^2}{4} \frac{(\lambda^2 + 1)}{(r - r_-)(r - r_+)} dr^2, \\ r_- = -\frac{\ell^2}{2} \left(\sqrt{a_0^2 - 4a_- a_+} - 2a_L \right), \quad r_+ = \frac{\ell^2}{2} \left(\sqrt{a_0^2 - 4a_- a_+} + 2a_L \right). \quad (\text{A.2})$$

We see that the two horizon radius of $\text{WAdS}_{(u,v,r)}$ are controlled by the way we do quotient. The metric can be transformed into the form of (2.25) with its two temperatures determined by r_+ and r_- . Under the choice of (4.8), (A.2) is already in the form of (2.25).

The asymptotic behaviour of this coordinate transformation

$$u = \frac{2 \tanh^{-1} \left(\frac{a_0 - 2a_+ U}{\sqrt{a_0^2 - 4a_- a_+}} \right)}{\sqrt{a_0^2 - 4a_- a_+}} + \mathcal{O} \left(\frac{1}{\rho^2} \right), \\ v = V - \frac{2a_L \tanh^{-1} \left(\frac{a_0 - 2a_+ U}{\sqrt{a_0^2 - 4a_- a_+}} \right)}{\sqrt{a_0^2 - 4a_- a_+}} + \mathcal{O} \left(\frac{1}{\rho} \right). \quad (\text{A.3})$$

indicate that the boundary of $\text{WAdS}_{(u,v,r)}$ covers a strip parallel to the U axis on the boundary of $\text{WAdS}_{(U,V,\rho)}$, with $a_0/(2a_+)$ controlling the center position of the strip, while $l_U = -\sqrt{a_0^2 - 4a_- a_+}/a_+$ controlling the width of the strip.

Consider the regulated interval (4.14) in the main text, its image in terms of the (u, v) coordinates is just (4.15) with

$$\Delta u = 2 \frac{\ell^2 \log \left(\frac{l_U}{\epsilon} \right)}{r_+ - r_-}, \\ \Delta v = l_V - 2a_L \frac{\ell^2 \log \left(\frac{l_U}{\epsilon} \right)}{r_+ - r_-}. \quad (\text{A.4})$$

As in the main text, we identify the end points and get a spacial circle $(u, v) \sim (u + \Delta u, v + \Delta v)$. We Integrate along the spacial circle at $r = r_+$, then get the thermal entropy for $\text{WAdS}_{(u,v,r)}$

$$S_{bh} = \frac{1}{4G} \int_0^1 \sqrt{g_{uu} \Delta u^2 + 2g_{uv} \Delta u \Delta v + g_{vv} \Delta v^2} d\tau \\ = \frac{1}{4G} \frac{(\ell^2 \Delta v + r_+ \Delta u)}{\ell} \\ = \frac{\ell}{4G} \log \frac{l_U}{\epsilon} + \frac{\ell}{4G} l_V. \quad (\text{A.5})$$

It is then obvious that, among all the four quantities controlled by the way we do quotient, the Bekenstein-Hawking entropy for $\text{WAdS}_{(u,v,r)}$ only depend on the width of the strip. This justifies our choice in the main text.

B Rindler method in AdS₃ revisited

In this Appendix, we revisit Rindler method in AdS₃ with our new strategy by doing quotient. In Appendix B.1, we consider AdS₃ with Brown-Henneaux boundary conditions and give a derivation for the HRT formula [29] in this context (AdS₃/CFT₂)⁶. This generalize the derivation of [25] to the covariant version. Then in Appendix B.2 we consider AdS₃ with CSS boundary conditions, and find the holographic entanglement entropy agrees with the entanglement entropy of a WCFT.

According to the RT (or HRT) [23, 24, 29] formula, the holographic entanglement entropy is proportional to a co-dimension two extremal surface, which is determined by the bulk metric. This indicates that the holographic entanglement entropy should be independent of the asymptotic boundary conditions. Here we consider the same metric with different boundary conditions and show explicitly how the choice of boundary conditions will change the holographic entanglement entropy for Poincaré AdS₃. This suggests that at least caution should be taken in applying RT (or HRT) proposal for holography beyond AdS/CFT.

Generalization to general temperatures can be done by considering the further mapping from BTZ black holes to Poincaré AdS₃ [47], and then follow the steps of Sec. 4.2 in the main text, or Sec. 5.5 of [29].

B.1 Brown-Henneaux boundary conditions

For simplicity we consider the Poincaré AdS₃ spacetime

$$ds^2 = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + 2\rho dU dV \right). \quad (\text{B.1})$$

Under the Brown-Henneaux boundary conditions, Poincaré AdS₃ is conjectured to be dual to the vacuum state of a CFT₂. The following six Killing vectors

$$\begin{aligned} J_- &= \partial_U, & J_0 &= U\partial_U - \rho\partial_\rho, & J_+ &= U^2\partial_U - \frac{1}{2\rho}\partial_V - 2\rho U\partial_\rho, \\ \tilde{J}_- &= \partial_V, & \tilde{J}_0 &= V\partial_V - \rho\partial_\rho, & \tilde{J}_+ &= V^2\partial_V - \frac{1}{2\rho}\partial_U - 2\rho V\partial_\rho. \end{aligned} \quad (\text{B.2})$$

are also asymptotic Killing vectors. The normalization are chosen to satisfy the standard $SL(2, R) \times SL(2, R)$ algebra

$$\begin{aligned} [J_-, J_+] &= 2J_0, & [J_0, J_\pm] &= \pm J_\pm, \\ [\tilde{J}_-, \tilde{J}_+] &= 2\tilde{J}_0, & [\tilde{J}_0, \tilde{J}_\pm] &= \pm \tilde{J}_\pm. \end{aligned} \quad (\text{B.3})$$

Now we do a coordinate transformation to obtain a new coordinate system with two explicit $U(1)$ symmetries. Define

$$\begin{aligned} J &= a_0 J_0 + a_+ J_+ + a_- J_-, \\ \tilde{J} &= \tilde{a}_0 \tilde{J}_0 + \tilde{a}_+ \tilde{J}_+ + \tilde{a}_- \tilde{J}_-, \end{aligned} \quad (\text{B.4})$$

⁶Recently a more general derivation for the HRT formula is given in [46].

where $a_0, a_+, a_-, \tilde{a}_0, \tilde{a}_+, \tilde{a}_-$ are arbitrary constants. Then we define new coordinates (u, v, r) such that there are two explicit Killing vectors

$$\partial_u = J, \quad \partial_v = \tilde{J}. \quad (\text{B.5})$$

As a result the new metric in (u, v, r) should only depend on r . Following the strategy of Sec. 4.1.1, we can do the quotient on Poincaré AdS₃. We find that the quotient gives a BTZ black string with its boundary covers the causal development of an interval. And the six parameters $a_0, a_+, a_-, \tilde{a}_0, \tilde{a}_+, \tilde{a}_-$ together control six quantities. Among which four quantities characterize the position and extension of an interval (or its causal development), and the rest two describe the two temperatures of $\text{BTZ}_{(u,v,r)}$. The regularized thermal entropy of $\text{BTZ}_{(u,v,r)}$ in fact only depends on the extension of the interval. While any choice, the purpose of this special choice

We choose the six parameters

$$\begin{aligned} a_0 &= 0, & a_+ &= -\frac{2}{l_U}, & a_- &= \frac{l_U}{2}, \\ \tilde{a}_0 &= 0, & \tilde{a}_+ &= 4l_V^{-2}, & \tilde{a}_- &= 1. \end{aligned} \quad (\text{B.6})$$

As we will see later, this choice is convenient for comparison with the CSS boundary conditions. Similar to the steps in section 4, we find the following coordinate transformation

$$\begin{aligned} u &= \frac{1}{4} \log \left[\frac{(1 + \rho(2U + l_U)V)^2 - \rho^2 l_V^2 (l_U/2 + U)^2}{(1 + \rho(2U - l_U)V)^2 - \rho^2 l_V^2 (l_U/2 - U)^2} \right], \\ v &= \frac{l_V}{4} \log \left[\frac{[(\rho(l_U - 2U)V - 1) + \rho l_V(l_U/2 - U)][(1 + \rho(l_U + 2U)V) + \rho l_V(l_U/2 + u)]}{[(\rho(l_U - 2U)V - 1) - \rho l_V(l_U/2 - U)][(1 + \rho(l_U + 2U)V) - \rho l_V(l_U/2 + u)]} \right], \\ r &= \frac{l_U \rho}{2} \left(1 - \frac{4V^2}{l_V^2} \right) - \frac{2\rho U^2}{l_U} + \frac{2(1 + 2\rho UV)^2}{l_U l_V^2 \rho}, \end{aligned} \quad (\text{B.7})$$

under which we get the metric of $\text{BTZ}_{(u,v,r)}$

$$ds^2 = \ell^2 \left(du^2 + 2r dudv + \frac{4}{l_V^2} dv^2 + \frac{dr^2}{4(r^2 - 4/l_V^2)} \right). \quad (\text{B.8})$$

The asymptotic behaviour of the coordinate transformations is given by

$$\begin{aligned} u &= \text{ArcTanh} \frac{2U}{l_U} + \mathcal{O} \left(\frac{1}{r} \right), \\ v &= \frac{l_V}{2} \text{ArcTanh} \left(\frac{2V}{l_V} \right) + \mathcal{O} \left(\frac{1}{r} \right), \end{aligned} \quad (\text{B.9})$$

which indicates the boundary of $\text{BTZ}_{(u,v,r)}$ (B.8) covers the causal development of an interval

$$\{(U, V) \mid -\frac{l_U}{2} < U < \frac{l_U}{2}, \quad -\frac{l_V}{2} < V < \frac{l_V}{2}\}. \quad (\text{B.10})$$

We can introduce two infinitesimal parameters ϵ_1 and ϵ_2 and regulate the interval as

$$\{(U, V) \mid U = (l_U - 2\epsilon_1)(-\frac{1}{2} + \tau), \quad V = (l_V - 2\epsilon_2)(-\frac{1}{2} + \tau), \quad \tau \in [0, 1]\}. \quad (\text{B.11})$$

The horizon of $\text{BTZ}_{(u,v,r)}$ can be depicted by

$$\begin{aligned} \{(u, v) | u = \Delta u(-\frac{1}{2} + \tau), \quad v = \Delta v(-\frac{1}{2} + \tau), \quad r = \frac{2}{l_V}, \quad \tau \in [0, 1]\}, \\ \Delta u = \log \frac{l_U}{\epsilon_1}, \quad \Delta v = \frac{l_V}{2} \log \frac{l_V}{\epsilon_2}. \end{aligned} \quad (\text{B.12})$$

Since Δu and Δv are infinite, as in the main text we identify the end points and get a spatial circle $(u, v) \sim (u + \Delta u, v + \Delta v)$, thus the Bekenstein-Hawking entropy of $\text{BTZ}_{(u,v,r)}$ (B.8) is given by

$$S_{\text{thermal}} = \frac{\ell}{4G} \log \frac{l_U l_V}{\epsilon_1 \epsilon_2}. \quad (\text{B.13})$$

Furthermore, to recover the entangling surface in the original vacuum AdS_3 spacetime, we can look at the inverse coordinate transformation

$$\begin{aligned} U &= \frac{l_U}{2} \frac{\sqrt{r^2 - 4l_V^{-2}} \sinh(u + 2v/l_V) - (r - 2l_V^{-1}) \sinh(u - 2v/l_V)}{\sqrt{r^2 - 4l_V^{-2}} \cosh(u + 2v/l_V) - (r - 2l_V^{-1}) \cosh(u - 2v/l_V)}, \\ V &= \frac{l_V}{2} \frac{\sqrt{r^2 - 4l_V^{-2}} \sinh(u + 2v/l_V) + (r - 2l_V^{-1}) \sinh(u - 2v/l_V)}{\sqrt{r^2 - 4l_V^{-2}} \cosh(u + 2v/l_V) - (r - 2l_V^{-1}) \cosh(u - 2v/l_V)}, \\ \rho &= \frac{\left(\sqrt{r^2 - 4l_V^{-2}} \cosh(u + 2v/l_V) - (r - 2l_V^{-1}) \cosh(u - 2v/l_V) \right)^2}{2l_U (r - 2l_V^{-1})}. \end{aligned} \quad (\text{B.14})$$

The image of the BTZ black hole horizon in AdS_3 is given by

$$U = \frac{l_U}{2} \tanh \tilde{\tau}, \quad V = \frac{l_V}{2} \tanh \tilde{\tau}, \quad \rho = \frac{2 \cosh^2 \tilde{\tau}}{l_U l_V}, \quad (\text{B.15})$$

where $\tilde{\tau}$ is defined by

$$\tilde{\tau} = u + 2v/l_V = \left(\tau - \frac{1}{2} \right) \log \frac{l_U l_V}{\epsilon_1 \epsilon_2}. \quad (\text{B.16})$$

One can check that (B.15) is just the geodesic ending on $(-\frac{l_U}{2}, -\frac{l_V}{2})$ and $(\frac{l_U}{2}, \frac{l_V}{2})$ on the AdS_3 boundary.

Note that when $\tau = 0, 1$, the geodesic goes to the two end points on the boundary with the same cutoff

$$\rho_{\text{max}} = \frac{1}{2\epsilon_1 \epsilon_2} + \frac{1}{l_U l_V} + \mathcal{O}(\epsilon^2). \quad (\text{B.17})$$

This indicates that the cutoff of the theory should be $\epsilon^2 = \epsilon_1 \epsilon_2$, and hence

$$S_{\text{thermal}} = \frac{\ell}{4G} \log \frac{l_U l_V}{\epsilon^2} \quad (\text{B.18})$$

which agrees with the HRT formula.

B.2 CSS (Dirichlet-Neumann) boundary conditions

For comparison, in this subsection we impose the CSS boundary conditions on AdS_3 , hence the dual field theory is conjectured to be a WCFT [18] (see Sec. 2.2.1). To compare with the previous subsection, we also consider Poincaré AdS_3 . All the calculations are the same as in section 4.2 with $T_V = T_U = \lambda = 0$. Below we will list the main differences between AdS/CFT and AdS/WCFT .

The first difference is the choice of Killing vectors. Under the CSS boundary conditions, \tilde{J}_+ and \tilde{J}_0 , though still isometry generators, are no longer asymptotic Killing vectors. Hence we need to set $\tilde{a}_0 = \tilde{a}_+ = 0$ to do the quotient. We have four parameters to do quotient under CSS, while six under Brown-Henneaux. As in section 4.1, we choose

$$a_0 = 0, \quad a_+ = -\frac{2}{l_U}, \quad a_- = \frac{l_U}{2}, \quad \tilde{a}_- = 1. \quad (\text{B.19})$$

Then the quotient is just the case in Sec. 4.2 with $\lambda = 0$, $T_U = T_V = 0$, and we get a $\text{BTZ}_{(u,v,r)}$.

The second difference is that l_V in this case is a free parameter rather than controlled by the way we do quotient. Under CSS, we can chose l_U and l_V to be the same as in Appendix B.1, while the thermal entropy of $\text{BTZ}_{(u,v,r)}$ is given by (4.33) with $T_V = T_U = 0$,

$$S_{\text{thermal}} = \frac{\ell}{4G} \log \frac{l_U}{\epsilon}. \quad (\text{B.20})$$

The disagreement between (B.18) and (B.20), which give the holographic entanglement entropy for the same interval while under different boundary conditions, is of course a result of the changing of boundary conditions.

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